

Kripke models and semantics: the intuition

A Kripke *model* (relational model) can be roughly pictured as a graph with labelled edges and nodes: there are some nodes (possible worlds, states, situations etc.), with some relations among them labelled by some symbols (modal operators), and each node is labelled by a set of symbols (proposition letters). A *frame* is a model without labels on nodes.

ϕ is *necessarily* true at the current (real) world iff ϕ is true at all the possible alternatives of the current world.

$\Box\phi$ formula holds at a world w iff for all the worlds v that are *reachable* from w : ϕ is true.

Kripke frames and models

“A model for *some language*”

The modal language based on a set \mathbf{P} of proposition letters and a *modal similarity type* $\tau = (\mathbf{O}, \rho)$ is as follows ($\mathbf{ML}(\mathbf{P}, \tau)$):

$$\phi ::= \top \mid p \mid \phi \wedge \phi \mid \neg\phi \mid \underbrace{\nabla(\phi \dots \phi)}_{\rho(\nabla)}$$

where $p \in \mathbf{P}$ and $\nabla \in \mathbf{O}$. (What if $\rho(\nabla)$ is 0?)

We define \perp as $\neg\top$, $\phi \rightarrow \psi$ as $\neg\phi \vee \psi$, and $\Delta(\phi_1 \dots \phi_{\rho(\nabla)})$ as $\neg\nabla(\neg\phi_1 \dots \neg\phi_{\rho(\nabla)})$.

When \mathbf{P} is clear in the context, we use \mathbf{ML} to denote the basic modal language with a single unary modality \square .

Kripke frames and models

Definition (Kripke frame and Kripke model)

A *Kripke frame* for language $\mathbf{ML}(\mathbf{P}, (\mathbf{O}, \rho))$ is a pair:

$$\mathcal{F} = \langle W, \{R_{\nabla} \mid \nabla \in \mathbf{O}\} \rangle$$

where:

- W is a *non-empty* set;
- $R_{\nabla} \subseteq \underbrace{W \times \cdots \times W}_{\rho(\nabla)+1}$ is a $\rho(\nabla) + 1$ -ary relation on W .

Since \mathcal{F} is not about \mathbf{P} we also call such \mathcal{F} a τ -frame given a similarity type τ . A *Kripke model* \mathcal{M} for $\mathbf{ML}(\mathbf{P}, \tau)$ is a pair $\langle \mathcal{F}, V \rangle$ where \mathcal{F} is a τ -frame and $V : W \rightarrow 2^{\mathbf{P}}$ is a *valuation function*. A *pointed Kripke model* (\mathcal{M}, w) is a Kripke model with a *designated point* w in the set of states of \mathcal{M} .

Kripke Semantics

Recall the language $\mathbf{ML}(\mathbf{P}, (\mathbf{O}, \rho))$:

$$\phi ::= \top \mid p \mid \phi \wedge \phi \mid \neg\phi \mid \underbrace{\nabla(\phi \dots \phi)}_{\rho(\nabla)}$$

Pay attention to the quantifiers in the last clause:

Kripke Semantics ($\rho(\nabla) > 0$)

$$\mathcal{M}, w \models \top \Leftrightarrow \text{always}$$

$$\mathcal{M}, w \models p \Leftrightarrow p \in V_{\mathcal{M}}(w)$$

$$\mathcal{M}, w \models \neg\phi \Leftrightarrow \mathcal{M}, w \not\models \phi$$

$$\mathcal{M}, w \models \phi \wedge \psi \Leftrightarrow \mathcal{M}, w \models \phi \text{ and } \mathcal{M}, w \models \psi$$

$$\begin{aligned} \mathcal{M}, w \models \nabla(\phi_1 \dots \phi_{\rho(\nabla)}) &\Leftrightarrow \forall w_1 \dots w_{\rho(\nabla)} : (\langle w, w_1, \dots, w_{\rho(\nabla)} \rangle \in R_{\nabla} \\ &\implies \exists i \in [1, \rho(\nabla)] : \mathcal{M}, w_i \models \phi_i) \end{aligned}$$

Kripke models of unary similarity types

Given $\tau = (\mathbf{O}, \rho)$, if for all $\nabla \in \mathbf{O}$: $\rho(\nabla) = 1$ then we call τ a *unary similarity type*. In such cases we often use indexed boxes (e.g., \Box_a) to denote modalities and write $w \xrightarrow{a} v$ for $(w, v) \in R_a$ in the frames and models. We say a model is of a unary similarity type if its frame is of a unary similarity type. Given \mathcal{M} , we write $W_{\mathcal{M}}$, $\xrightarrow{a}_{\mathcal{M}}$ and $V_{\mathcal{M}}$ for the corresponding components in \mathcal{M} .

$\mathcal{M}, w \vDash \Box_a \phi$ iff for all $v : w \xrightarrow{a} v \implies \mathcal{M}, v \vDash \phi$

$\mathcal{M}, w \vDash \neg \Box_a \neg \phi$ iff it is not the case that $(\forall v : w \xrightarrow{a} v \implies \mathcal{M}, v \vDash \neg \phi)$

$\mathcal{M}, w \vDash \Diamond_a \phi$ iff there exists $v : w \xrightarrow{a} v$ and $\mathcal{M}, v \vDash \phi$

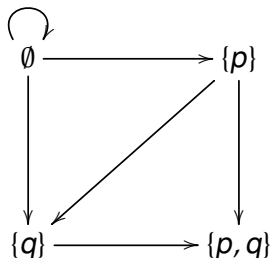
Some properties of Kripke models

A Kripke frame \mathcal{M} of a unary similarity type is said to be:

- *finite*, if $W_{\mathcal{M}}$ is finite. (In CS the relations are also required to be finitely representable.)
- *image-finite*, if for any world $w \in W_{\mathcal{M}}$, any $a \in \mathbf{O}$, $\{v \mid w \xrightarrow{a} v\}$ is finite.
- *deterministic*, if for any world $w \in W$, any $a \in \mathbf{O}$, $w \xrightarrow{a} v$ and $w \xrightarrow{a} v'$ implies $v = v'$.
- *well-founded*, if there is no infinite upward sequence:
 $\dots \xrightarrow{a_2} w_2 \xrightarrow{a_1} w_1 \xrightarrow{a_0} w_0.$
- *acyclic*, if there is no cycles.
- \mathcal{M}, w is a *tree* if \mathcal{M} is acyclic, w can reach all the other nodes, and each non- w node has a unique predecessor.

An example

Consider the following Kripke model \mathcal{M} of the basic similarity type $(\tau = (\{\Box\}, \rho)$ and $\rho(\Box) = 1)$:



Where is $\Box\Diamond p$ true? Where is $\Box\Diamond p \wedge \Diamond\Box p$ true?

Model checking problems

Two model checking problems:

1. Local model checking: testing whether $\mathcal{M}, w \models \phi$;
2. Global model checking: compute the set

$$\{w \in W_{\mathcal{M}} \mid \mathcal{M}, w \models \phi\}$$

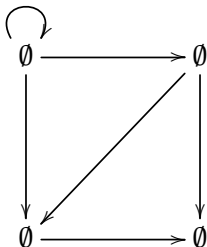
Let $I_R(X) = \{w \in W_{\mathcal{M}} \mid \forall v : w \rightarrow v \implies v \in X\}$, we can define the *extension* of formulas in \mathcal{M} :

$$\begin{aligned} \llbracket \top \rrbracket^{\mathcal{M}} &= W_{\mathcal{M}} & \llbracket p \rrbracket^{\mathcal{M}} &= \{w \mid p \in V(w)\} \\ \llbracket \neg \phi \rrbracket^{\mathcal{M}} &= W \setminus \llbracket \phi \rrbracket^{\mathcal{M}} & \llbracket \phi \wedge \psi \rrbracket^{\mathcal{M}} &= \llbracket \phi \rrbracket^{\mathcal{M}} \cap \llbracket \psi \rrbracket^{\mathcal{M}} \\ \llbracket \Box \phi \rrbracket^{\mathcal{M}} &= I_R \llbracket \phi \rrbracket^{\mathcal{M}} \end{aligned}$$

An algorithm for global model checking ϕ on \mathcal{M} : labelling the states of \mathcal{M} by the sub-formulas of ϕ (according to their complexity) that are true at each state.

An example

Consider the following Kripke model \mathcal{M} of the basic similarity type $(\tau = \langle \{\Box\}, \rho \rangle$ and $\rho(\Box) = 1)$:



Where is $\Diamond\Box p$ true? Where is $\Box\Diamond p \wedge \Diamond\Box p$ true? Can we characterize each state by a modal formula?

Satisfiability and Validity

Modal formulas can talk about models or frames at local or global levels, this induces 4 cases:

ϕ is *satisfiable* at \mathcal{M}, w if $\mathcal{M}, w \vDash \phi$.

ϕ is *globally satisfiable* (or valid) in \mathcal{M} ($\mathcal{M} \vDash \phi$) if for all w in \mathcal{M} : $\mathcal{M}, w \vDash \phi$.

ϕ is valid in a pointed frame \mathcal{F}, w ($\mathcal{F}, w \vDash \phi$) if for any model based on \mathcal{F}, w : $\mathcal{M}, w \vDash \phi$.

ϕ is valid in a frame \mathcal{F} ($\mathcal{F} \vDash \phi$) if for any model based on \mathcal{F} : $\mathcal{M} \vDash \phi$.

Similarly, we can define the validity of a formula w.r.t. classes of models or frames.

When can two pointed models be seen as the same thing?

What do you think? It depends on the measurement...

- Structural equivalences (\approx_S), e.g., isomorphism.
- Logical equivalences (\equiv_L) (two models cannot be told apart by any formula of a certain logical language L).
- Matching the two: logical characterization of the structural equivalence: $\mathcal{M}, w \equiv_L \mathcal{N}, v \iff \mathcal{M}, w \approx_S \mathcal{N}, v$.

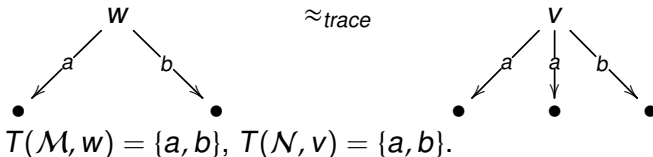
If $\mathcal{M}, w \approx_S \mathcal{N}, v$ implies $\mathcal{M}, w \equiv_L \mathcal{N}, v$ then we say L is *invariant* under \approx_S (or \approx_S is an *invariance* for L).

Equivalence notions: examples

For simplicity, we ignore the valuation first.

Trace equivalence

Let $T(\mathcal{M}, w) = \{s \mid s = a_0 \dots a_k \text{ and there is a path } w \xrightarrow{a_0} w_0 \xrightarrow{a_1} \dots \xrightarrow{a_k} w_k \text{ in } \mathcal{M}\}$. $\mathcal{M}, w \approx_{\text{trace}} \mathcal{N}, v \iff T(\mathcal{M}, w) = T(\mathcal{N}, v)$.



Q: Which syntactic fragment of modal logic can characterize this equivalence notion? (Hint: we only need formulas like $\diamond_a \dots \diamond_{a'} \top$)

Equivalence notions via behaviours

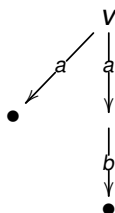
Complete trace equivalence

$CT(\mathcal{M}, w) = \{s \mid s = a_0 \dots a_k \text{ and there is a path } w \xrightarrow{a_0} w_0 \xrightarrow{a_1} \dots \xrightarrow{a_k} w_k \text{ in } \mathcal{M} \text{ such that } w_k \text{ does not have successors}\}$

$\mathcal{M}, w \approx_{ctrace} \mathcal{N}, v \iff (T(\mathcal{M}, w) = T(\mathcal{N}, v) \text{ and } CT(\mathcal{M}, w) = CT(\mathcal{N}, v)).$



\approx_{trace} but $\not\approx_{ctrace}$



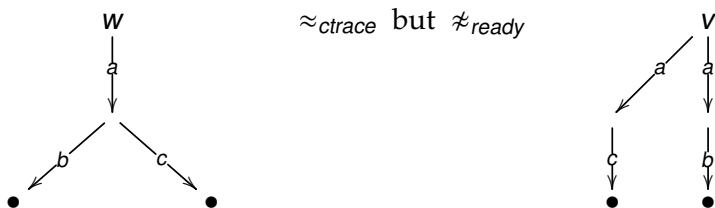
$CT(\mathcal{M}, w) = \{ab\}, CT(\mathcal{N}, v) = \{a, ab\}$

Equivalence notions

Readiness equivalence

Let $R(\mathcal{M}, w) = \{(s, A) \mid s = a_0 \dots a_k, \text{ and there is a path } w \xrightarrow{a_0} w_0 \xrightarrow{a_1} \dots \xrightarrow{a_k} w_k \text{ in } \mathcal{M} \text{ and } A = \{a \mid w_k \xrightarrow{a} v\}\}$.

$\mathcal{M}, w \approx_{\text{readiness}} \mathcal{N}, v \iff R(\mathcal{M}, w) = R(\mathcal{N}, v)$.



$R(\mathcal{M}, w) = \{(a, \{b, c\}), (ab, \emptyset), (ac, \emptyset)\}$

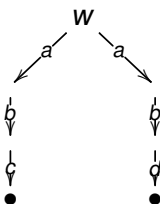
$R(\mathcal{N}, v) = \{(a, \{c\}), (a, \{b\}), (ac, \emptyset), (ab, \emptyset)\}$

Equivalence notions

Possible future equivalence

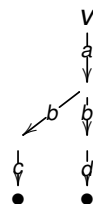
Let $P(\mathcal{M}, w) = \{(s, X) \mid s = a_0 \dots a_k \text{ and there is a path } w \xrightarrow{a_0} w_0 \xrightarrow{a_1} \dots \xrightarrow{a_k} w_k \text{ in } \mathcal{M} \text{ and } X = T(\mathcal{M}, w_k)\}$.

$\mathcal{M}, w \approx_{PF} \mathcal{N}, v \iff P(\mathcal{M}, w) = P(\mathcal{N}, v)$. Q: can you characterize \approx_{PF} by a fragment of modal logic?



$$P(\mathcal{M}, w) = \{(a, \{b, bc\}), (a, \{b, bd\}), (ab, \{c\}), (ab, \{d\}), (abc, \emptyset), (abd, \emptyset)\}$$

\approx_{ready} but $\not\approx_{pf}$



$$P(\mathcal{N}, v) = \{(a, \{b, bc, bd\}), (ab, \{c\}), (ab, \{d\}), (abc, \emptyset), (abd, \emptyset)\}$$

Equivalence notions

The above notions can be lifted to incorporate valuations.

Clearly $\approx_{PF} \subseteq \approx_{ready} \subseteq \approx_{ctrace} \subseteq \approx_{trace}$.

We say \approx is *finer* than \approx' (or \approx' is *coarser* than \approx) if $\approx \subseteq \approx'$.

We need fragments of modal logic which have stronger distinguishing power to characterize finer structural equivalences than the coarser ones.

There are (finite) models that are PF-equivalent but still can be distinguished by a modal formula of basic similarity type (Exercise). Then the question is: what is the structure equivalence notion corresponding to modal equivalence?

Further reading

B. Jack Copeland: The Genesis of Possible Worlds Semantics. *J. Philosophical Logic* 31(2): 99-137 (2002)

B. Jack Copeland: Meredith, Prior, and the History of Possible Worlds Semantics. *Synthese* 150(3): 373-397 (2006)