FORMAL METHODS

LECTURE IV: COMPUTATION TREE LOGIC (CTL)

Alessandro Artale

Faculty of Computer Science – Free University of Bolzano

`artale@inf.unibz.it` `http://www.inf.unibz.it/~artale/`

Some material (text, figures) displayed in these slides is courtesy of:
M. Benerecetti, A. Cimatti, M. Fisher, F. Giunchiglia, M. Pistore, M. Roveri, R.Sebastiani.

Summary of Lecture IV

- Computation Tree Logic: Intuitions.
- CTL: Syntax and Semantics.
- CTL in Computer Science.
- CTL and Model Checking: Examples.
- CTL Vs. LTL.
- CTL*.

Computation Tree logic Vs. LTL

- LTL implicitly quantifies *universally* over paths.

  $\langle \mathcal{KM}, s\rangle \models \phi$ iff for every path π starting at s $\langle \mathcal{KM}, \pi\rangle \models \phi$

- Properties that assert the *existence* of a path cannot be expressed. In particular, properties which *mix* existential and universal path quantifiers cannot be expressed.
- The *Computation Tree Logic*, CTL, solves these problems!
  - CTL explicitly introduces *path quantifiers*!
  - CTL is the natural temporal logic interpreted over Branching Time Structures.

CTL at a glance

- CTL is evaluated over branching-time structures (Trees).
- CTL explicitly introduces *path quantifiers*:

  All Paths: $\boxed{\mathrm{P}}$

  Exists a Path: $\langle\mathrm{P}\rangle$.

- Every temporal operator $(\Box, \Diamond, \bigcirc, \mathcal{U})$ preceded by a path quantifier ($\boxed{\mathrm{P}}$ or $\langle\mathrm{P}\rangle$).
- **Universal modalities:** $\boxed{\mathrm{P}}\Diamond, \boxed{\mathrm{P}}\Box, \boxed{\mathrm{P}}\bigcirc, \boxed{\mathrm{P}}\mathcal{U}$
  The temporal formula is true in **all** the paths starting in the current state.
- **Existential modalities:** $\langle\mathrm{P}\rangle\Diamond, \langle\mathrm{P}\rangle\Box, \langle\mathrm{P}\rangle\bigcirc, \langle\mathrm{P}\rangle\mathcal{U}$
  The temporal formula is true in **some** path starting in the current state.

Summary

- Computation Tree Logic: Intuitions.
- CTL: Syntax and Semantics.
- CTL in Computer Science.
- CTL and Model Checking: Examples.
- CTL Vs. LTL.
- CTL*.

CTL: Syntax

Countable set Σ of *atomic propositions*: $p, q, \ldots$ the set $\textsc{Form}$ of formulas is:

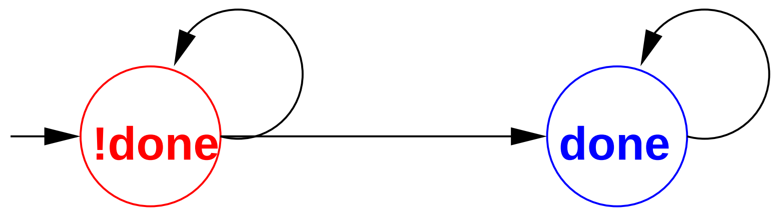
$$\varphi, \psi \quad \longrightarrow \quad p \mid \top \mid \bot \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid$$

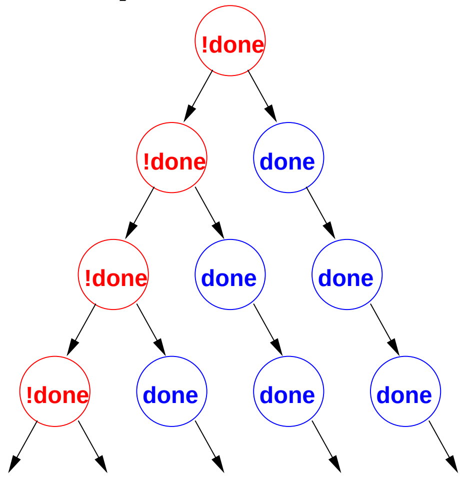
$$\boxed{\mathrm{P}}\ \bigcirc\varphi \mid \boxed{\mathrm{P}}\ \Box\varphi \mid \boxed{\mathrm{P}}\ \Diamond\varphi \mid \boxed{\mathrm{P}}\ (\varphi\ \mathcal{U}\ \psi)$$

$$\langle\mathrm{P}\rangle\ \bigcirc\varphi \mid \langle\mathrm{P}\rangle\ \Box\varphi \mid \langle\mathrm{P}\rangle\ \Diamond\varphi \mid \langle\mathrm{P}\rangle\ (\varphi\ \mathcal{U}\ \psi)$$

CTL: Semantics

- We interpret our CTL temporal formulas over Kripke Models linearized as trees (e.g. $\boxed{\mathrm{P}}\Diamond done$).





- Universal modalities ($\boxed{\mathrm{P}}\Diamond$, $\boxed{\mathrm{P}}\Box$, $\boxed{\mathrm{P}}\bigcirc$, $\boxed{\mathrm{P}}\,\mathcal{U}$): the temporal formula is true in **all** the paths starting in the current state.
- Existential modalities ($\langle\mathrm{P}\rangle\Diamond$, $\langle\mathrm{P}\rangle\Box$, $\langle\mathrm{P}\rangle\bigcirc$, $\langle\mathrm{P}\rangle\,\mathcal{U}$): the temporal formula is true in **some** path starting in the current state.

CTL: Semantics (Cont.)

Let Σ be a set of atomic propositions. We interpret our CTL temporal formulas over Kripke Models:

$$\mathcal{KM} \;=\; \langle S, I, R, \Sigma, L \rangle$$

The semantics of a temporal formula is provided by the *satisfaction* relation:

$$\models : (\mathcal{KM} \times S \times \textsc{Form}) \rightarrow \{\textbf{true}, \textbf{false}\}$$

CTL Semantics: The Propositional Aspect

We start by defining when an atomic proposition is true at a state/time "s_i"

$$\mathcal{KM}, s_i \models p \quad \textbf{iff} \quad p \in L(s_i) \qquad \text{(for } p \in \Sigma\text{)}$$

The semantics for the classical operators is as expected:

$$\mathcal{KM}, s_i \models \neg\varphi \quad \textbf{iff} \quad \mathcal{KM}, s_i \not\models \varphi$$

$$\mathcal{KM}, s_i \models \varphi \wedge \psi \quad \textbf{iff} \quad \mathcal{KM}, s_i \models \varphi \text{ and } \mathcal{KM}, s_i \models \psi$$

$$\mathcal{KM}, s_i \models \varphi \vee \psi \quad \textbf{iff} \quad \mathcal{KM}, s_i \models \varphi \text{ or } \mathcal{KM}, s_i \models \psi$$

$$\mathcal{KM}, s_i \models \varphi \Rightarrow \psi \quad \textbf{iff} \quad \text{if } \mathcal{KM}, s_i \models \varphi \text{ then } \mathcal{KM}, s_i \models \psi$$

$$\mathcal{KM}, s_i \models \top$$

$$\mathcal{KM}, s_i \not\models \bot$$

CTL Semantics: The Temporal Aspect

Temporal operators have the following semantics where $\pi = (s_i, s_{i+1}, \ldots)$ is a generic path outgoing from state $s_i \mathrm{in} \mathcal{KM}$.

$$
\begin{array}{llll}
\mathcal{KM}, s_i \models \boxed{\mathrm{P}} \bigcirc \varphi & \text{iff} & \forall \pi = (s_i, s_{i+1}, \ldots) & \mathcal{KM}, s_{i+1} \models \varphi \\
\mathcal{KM}, s_i \models \langle \mathrm{P} \rangle \bigcirc \varphi & \text{iff} & \exists \pi = (s_i, s_{i+1}, \ldots) & \mathcal{KM}, s_{i+1} \models \varphi \\
\mathcal{KM}, s_i \models \boxed{\mathrm{P}} \Box \varphi & \text{iff} & \forall \pi = (s_i, s_{i+1}, \ldots) & \forall j \geq i. \mathcal{KM}, s_j \models \varphi \\
\mathcal{KM}, s_i \models \langle \mathrm{P} \rangle \Box \varphi & \text{iff} & \exists \pi = (s_i, s_{i+1}, \ldots) & \forall j \geq i. \mathcal{KM}, s_j \models \varphi \\
\mathcal{KM}, s_i \models \boxed{\mathrm{P}} \Diamond \varphi & \text{iff} & \forall \pi = (s_i, s_{i+1}, \ldots) & \exists j \geq i. \mathcal{KM}, s_j \models \varphi \\
\mathcal{KM}, s_i \models \langle \mathrm{P} \rangle \Diamond \varphi & \text{iff} & \exists \pi = (s_i, s_{i+1}, \ldots) & \exists j \geq i. \mathcal{KM}, s_j \models \varphi \\
\mathcal{KM}, s_i \models \boxed{\mathrm{P}} (\varphi \, \mathcal{U} \, \psi) & \text{iff} & \forall \pi = (s_i, s_{i+1}, \ldots) & \exists j \geq i. \mathcal{KM}, s_j \models \psi \text{ and} \\
 & & & \forall i \leq k < j \ : M, s_k \models \varphi \\
\mathcal{KM}, s_i \models \langle \mathrm{P} \rangle (\varphi \, \mathcal{U} \, \psi) & \text{iff} & \exists \pi = (s_i, s_{i+1}, \ldots) & \exists j \geq i. \mathcal{KM}, s_j \models \psi \text{ and} \\
 & & & \forall i \leq k < j \ : \mathcal{KM}, s_k \models \varphi
\end{array}
$$

CTL Semantics: Intuitions

CTL is given by the standard boolean logic enhanced with temporal operators.

- “Necessarily Next”. $\boxed{\mathrm{P}}\bigcirc\varphi$ is true in s_t iff φ is true in every successor state s_{t+1}
- “Possibly Next”. $\Diamond\!\!\!\!\!{\scriptstyle \mathrm{P}}\;\bigcirc\varphi$ is true in s_t iff φ is true in one successor state s_{t+1}
- “Necessarily in the future” (or “Inevitably”). $\boxed{\mathrm{P}}\Diamond\varphi$ is true in s_t iff φ is inevitably true in **some** $s_{t'}$ with $t' \geq t$
- “Possibly in the future” (or “Possibly”). $\Diamond\!\!\!\!\!{\scriptstyle \mathrm{P}}\;\Diamond\varphi$ is true in s_t iff φ may be true in **some** $s_{t'}$ with $t' \geq t$

CTL Semantics: Intuitions (Cont.)

- “Globally” (or “always”). $\boxed{\mathrm{P}}\ \Box\varphi$ is true in s_t iff φ is true in **all** $s_{t'}$ with $t' \geq t$
- “Possibly henceforth”. $\Diamond_{\mathrm{P}}\ \Box\varphi$ is true in s_t iff φ is possibly true henceforth
- “Necessarily Until”. $\boxed{\mathrm{P}}\ (\varphi\,\mathcal{U}\,\psi)$ is true in s_t iff necessarily φ holds until ψ holds.
- “Possibly Until”. $\Diamond_{\mathrm{P}}\ (\varphi\,\mathcal{U}\,\psi)$ is true in s_t iff possibly φ holds until ψ holds.

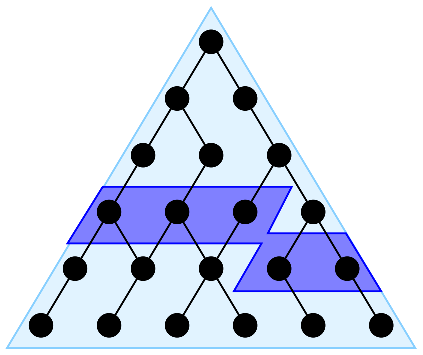
CTL Alternative Notation

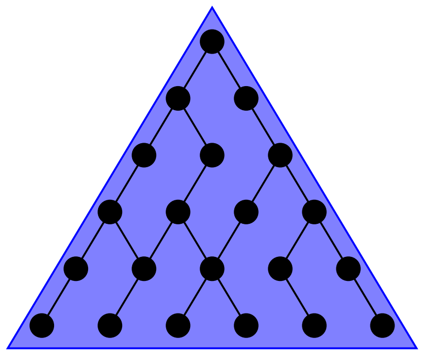
Alternative notations are used for temporal operators.

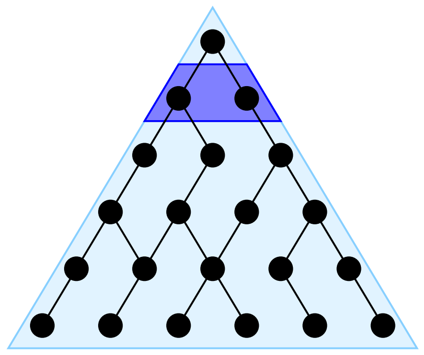
$\Diamond_{\mathrm{P}}$	$\leadsto$	E	there Exists a path
$\boxed{\mathrm{P}}$	$\leadsto$	A	in All paths
$\Diamond$	$\leadsto$	F	sometime in the Future
$\Box$	$\leadsto$	G	Globally in the future
$\bigcirc$	$\leadsto$	X	neXtime

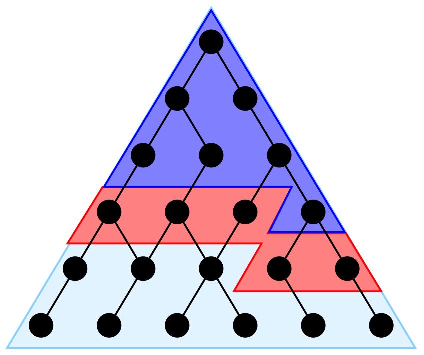
CTL Semantics: Intuitions (Cont.)

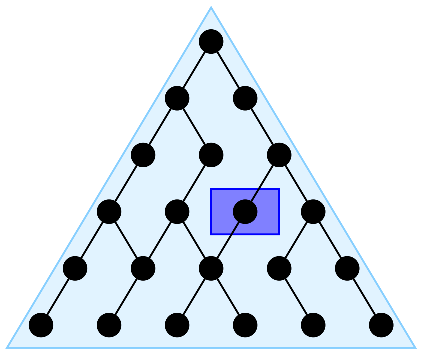
finally p | globally p | next p | p until q

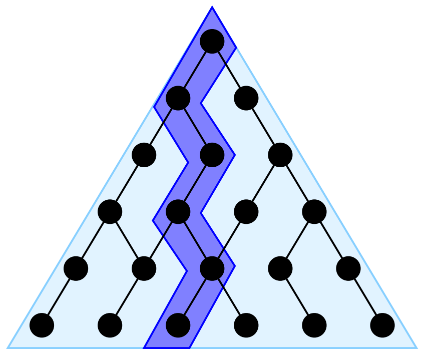

AF p

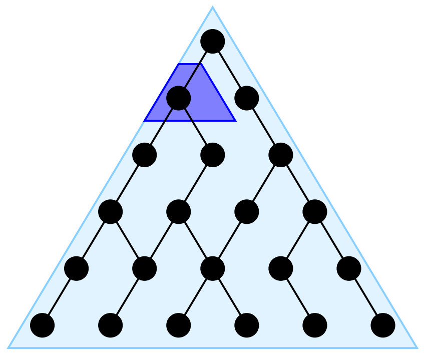

AG p

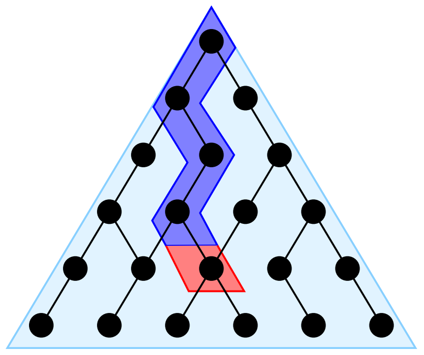

AX p


A[p U q]


EF p


EG p


EX p


E[p U q]

A Complete Set of CTL Operators

All CTL operators can be expressed via: $\langle P \rangle \bigcirc, \langle P \rangle \Box, \langle P \rangle \mathcal{U}$

- $[P] \bigcirc \equiv \neg \langle P \rangle \bigcirc \neg \varphi$
- $[P] \Diamond \varphi \equiv \neg \langle P \rangle \Box \neg \varphi$
- $\langle P \rangle \Diamond \varphi \equiv \langle P \rangle (\top \mathcal{U} \varphi)$
- $[P] \Box \varphi \equiv \neg \langle P \rangle \Diamond \neg \varphi \equiv \neg \langle P \rangle (\top \mathcal{U} \neg \varphi)$
- $[P] (\varphi \mathcal{U} \psi) \equiv \neg \langle P \rangle \Box \neg \psi \wedge \neg \langle P \rangle (\neg \psi \mathcal{U} (\neg \varphi \wedge \neg \psi))$

Summary

- Computation Tree Logic: Intuitions.
- CTL: Syntax and Semantics.
- CTL in Computer Science.
- CTL and Model Checking: Examples.
- CTL Vs. LTL.
- CTL*.

Safety Properties

Safety:

“something bad will not happen”

Typical examples:

$\boxed{\mathrm{P}}\ \Box\neg(reactor\_temp > 1000)$

$\boxed{\mathrm{P}}\ \Box\neg(one\_way \wedge \boxed{\mathrm{P}}\ \bigcirc other\_way)$

$\boxed{\mathrm{P}}\ \Box\neg((x = 0) \wedge \boxed{\mathrm{P}}\ \bigcirc \boxed{\mathrm{P}}\ \bigcirc \boxed{\mathrm{P}}\ \bigcirc(y = z/x))$

and so on.....

Usually: $\boxed{\mathrm{P}}\ \Box\neg\ldots.$

Liveness Properties

Liveness:

“something good will happen”

Typical examples:

$\boxed{\text{P}}$ $\Diamond rich$

$\boxed{\text{P}}$ $\Diamond(x > 5)$

$\boxed{\text{P}}$ $\Box(start \Rightarrow$ $\boxed{\text{P}}$ $\Diamond terminate)$

and so on.....

Usually: $\boxed{\text{P}}$ $\Diamond \ldots$

Fairness Properties

Often only really useful when scheduling processes, responding to messages, etc.

Fairness:

"something is successful/allocated infinitely often"

Typical example:

$$\boxed{\mathrm{P}}\ \square(\boxed{\mathrm{P}}\,\Diamond enabled)$$

Usually: $\boxed{\mathrm{P}}\ \square\ \boxed{\mathrm{P}}\,\Diamond\ldots$

Summary

- Computation Tree Logic: Intuitions.
- CTL: Syntax and Semantics.
- CTL in Computer Science.
- CTL and Model Checking: Examples.
- CTL Vs. LTL.
- CTL*.

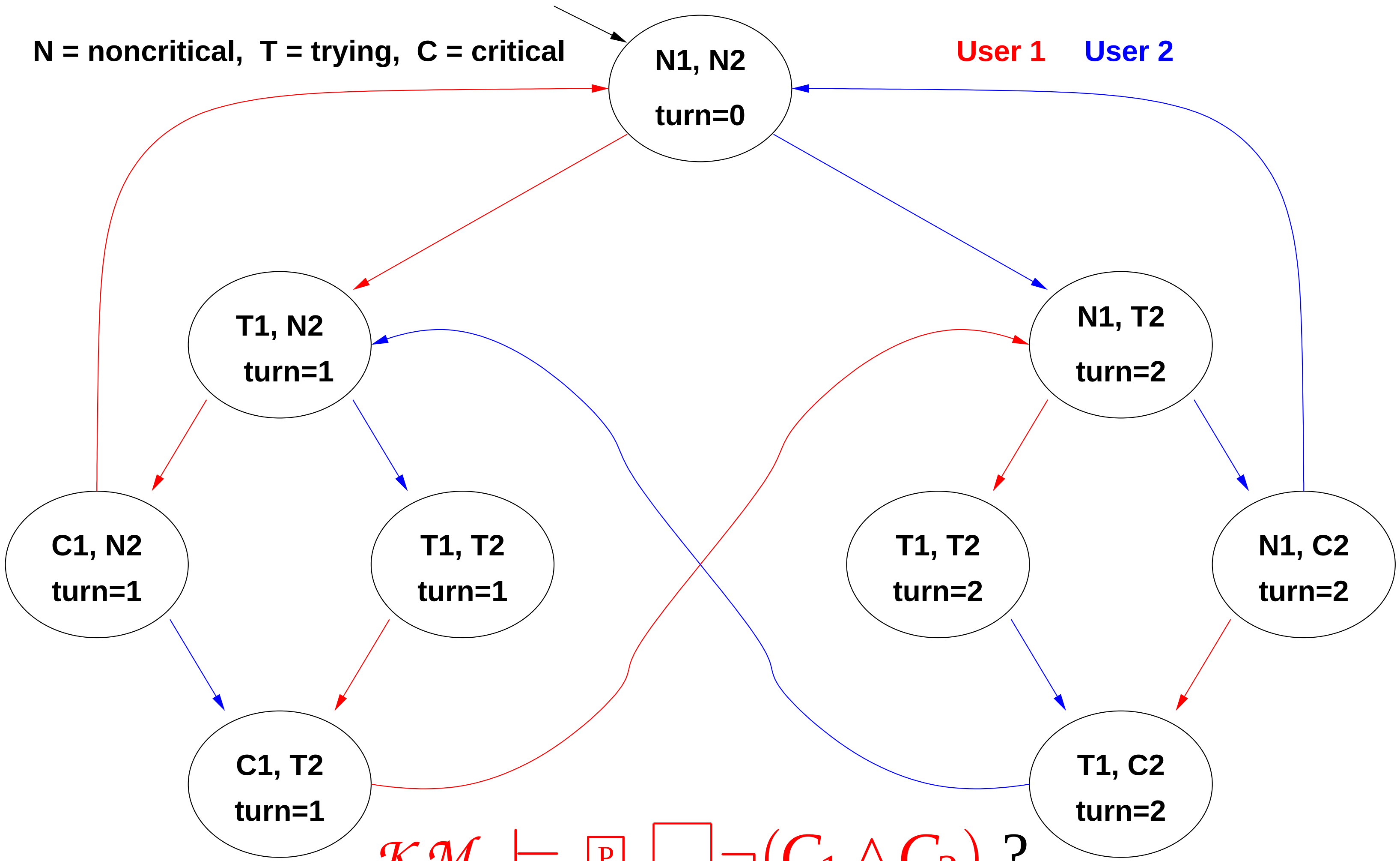
The CTL Model Checking Problem

The CTL Model Checking Problem is formulated as:

$$\mathcal{KM} \models \phi$$

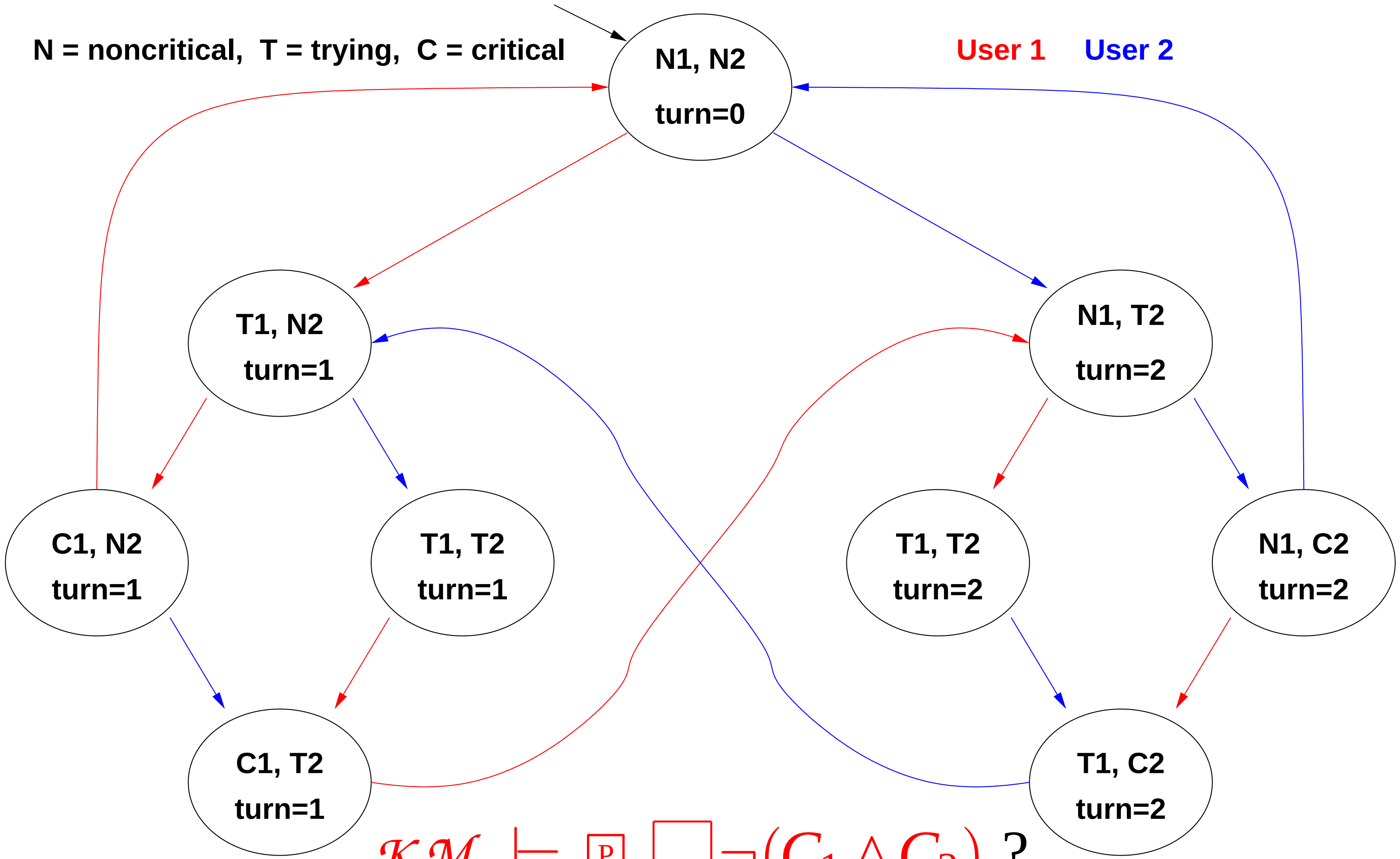
Check if $\mathcal{KM}, s_0 \models \phi$, for **every initial state**, s_0, of the Kripke structure $\mathcal{KM}$.

Example 1: Mutual Exclusion (Safety)



$$\mathcal{KM} \models \boxed{\text{P}}\ \Box \neg (C_1 \wedge C_2)\ ?$$

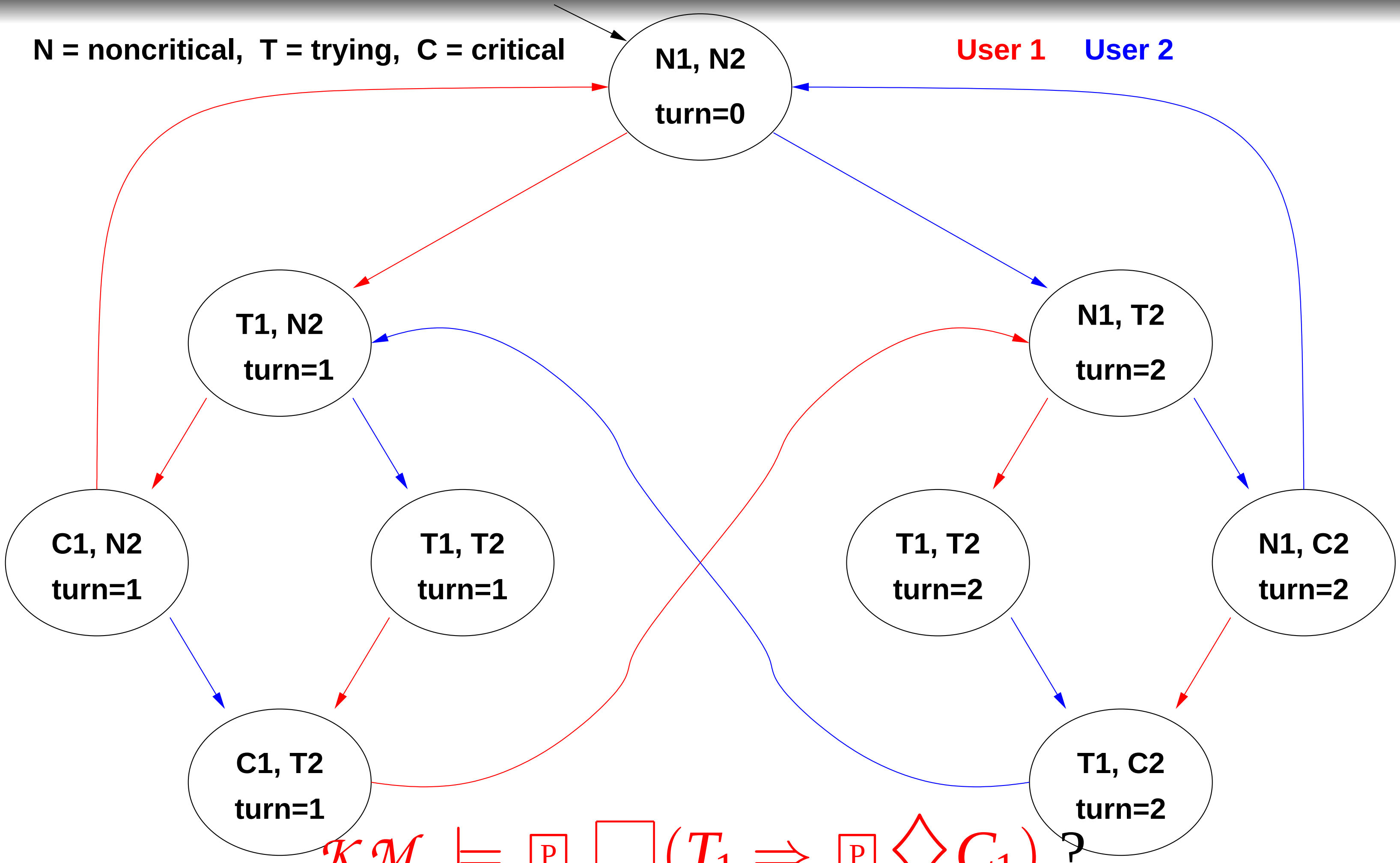
Example 1: Mutual Exclusion (Safety)



$\mathcal{KM} \models \boxed{\text{P}} \; \Box \neg (C_1 \wedge C_2)$?

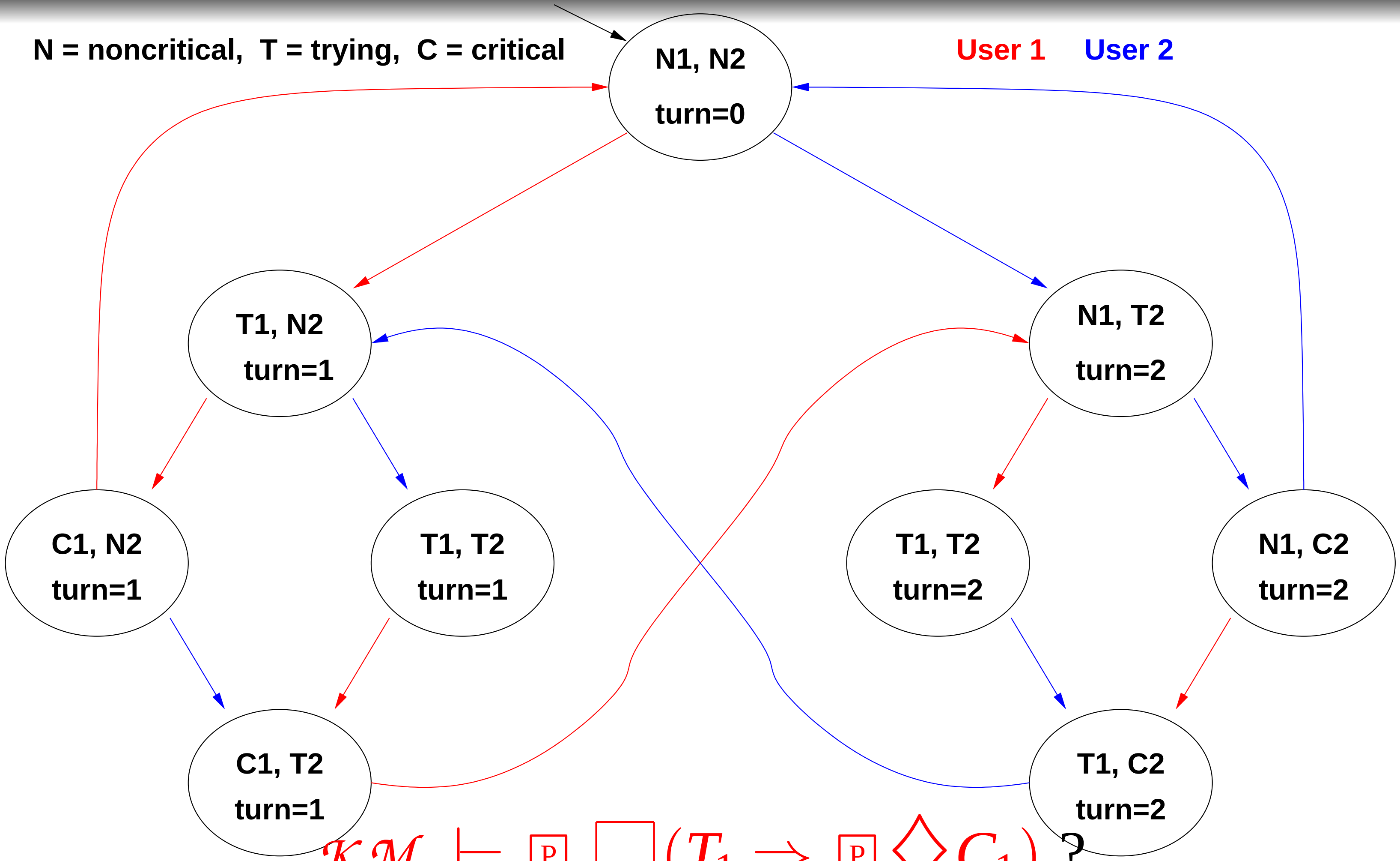
YES: There is no reachable state in which $(C_1 \wedge C_2)$ holds!
(Same as the $\Box \neg (C_1 \wedge C_2)$ in LTL.)

Example 2: Liveness



$\mathcal{KM} \models \boxed{P}\ \square(T_1 \Rightarrow \boxed{P}\ \Diamond C_1)$?

Example 2: Liveness

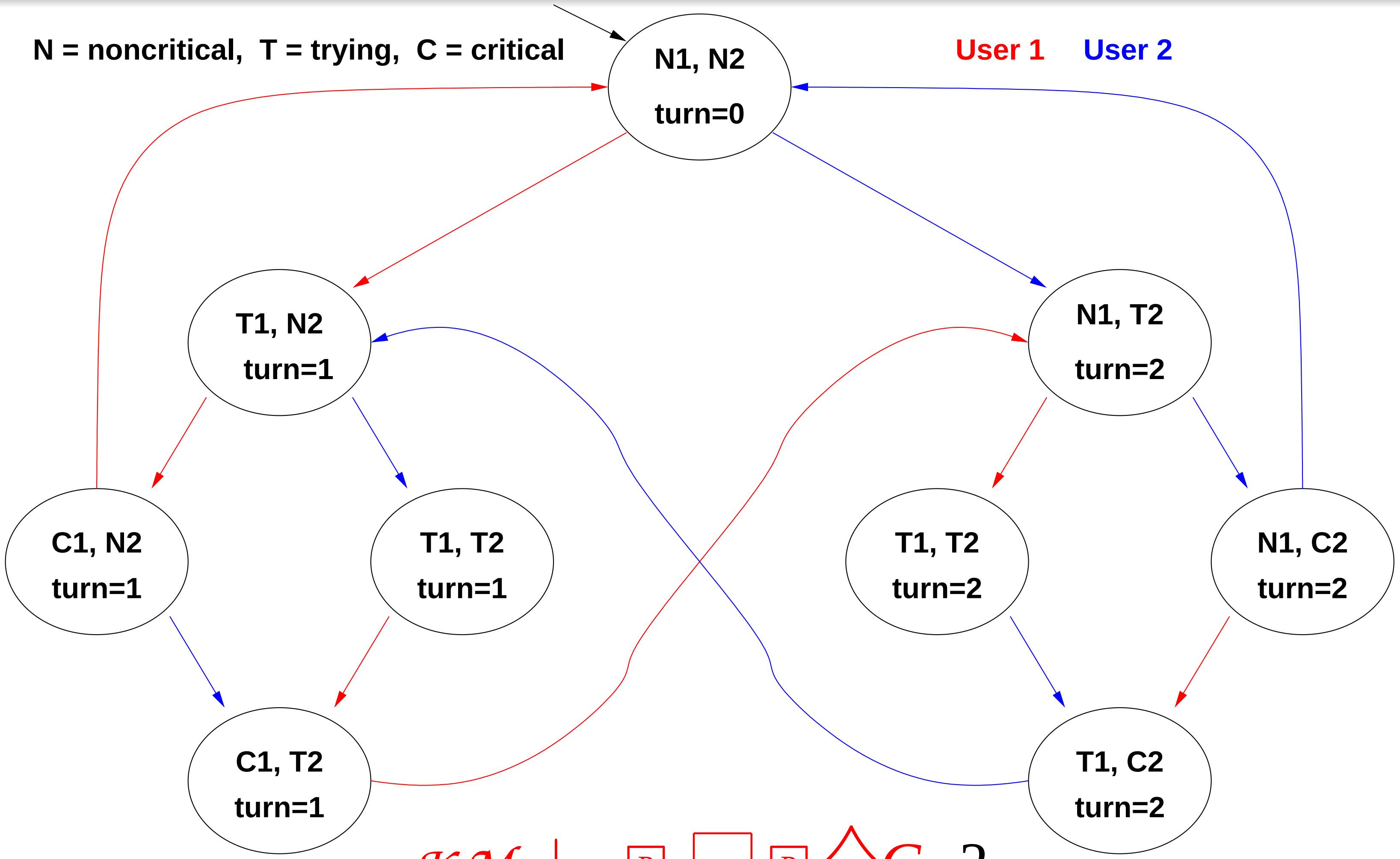


$\mathcal{KM} \models \boxed{\text{P}}\,\Box(T_1 \Rightarrow \boxed{\text{P}}\,\Diamond C_1)$?

YES: every path starting from each state where T_1 holds passes through a state where C_1 holds.

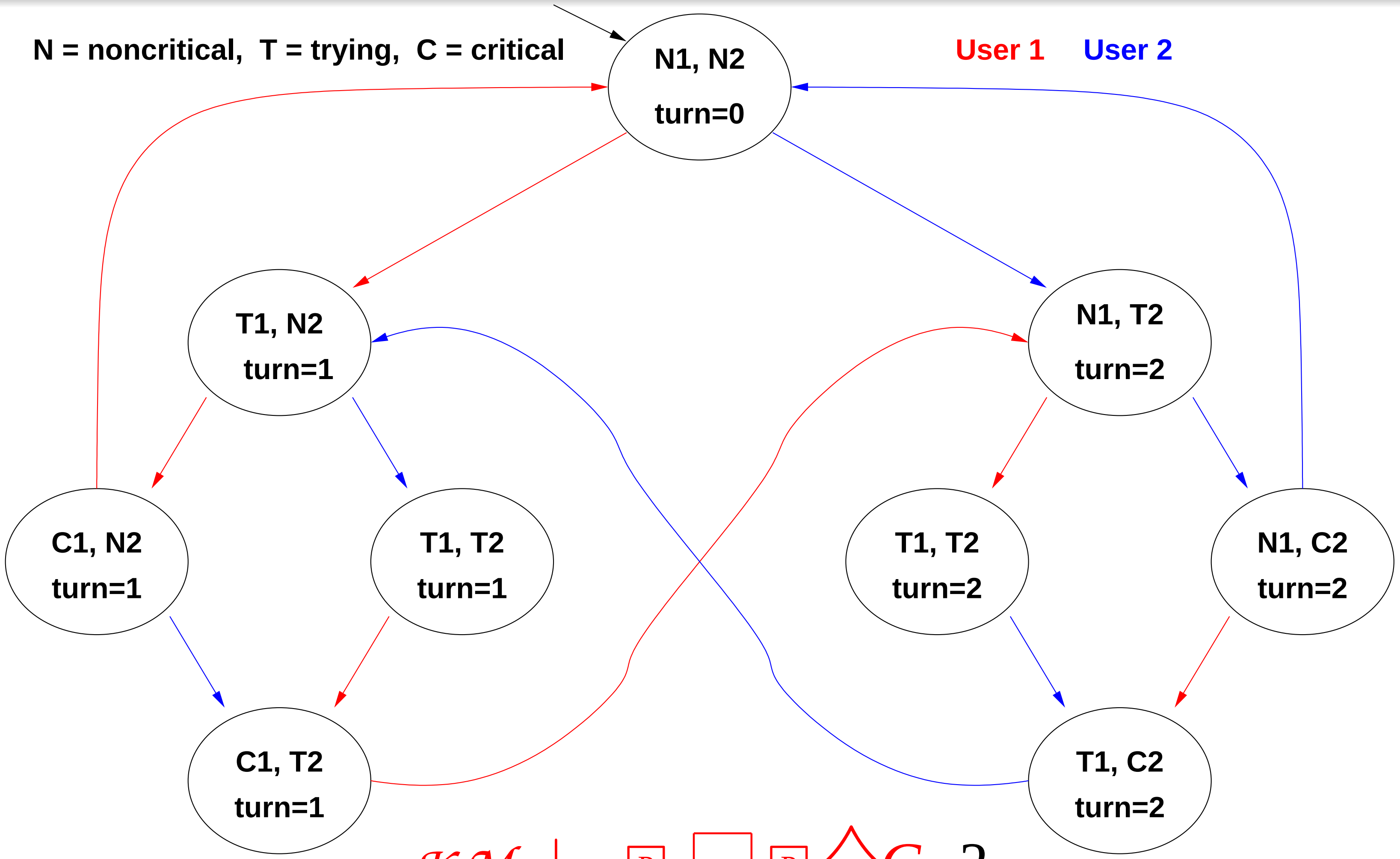
(Same as $\Box(T_1 \Rightarrow \Diamond C_1)$ in LTL)

Example 3: Fairness



$$\mathcal{KM} \models \boxed{\text{P}}\ \Box\ \boxed{\text{P}}\ \Diamond C_1\ ?$$

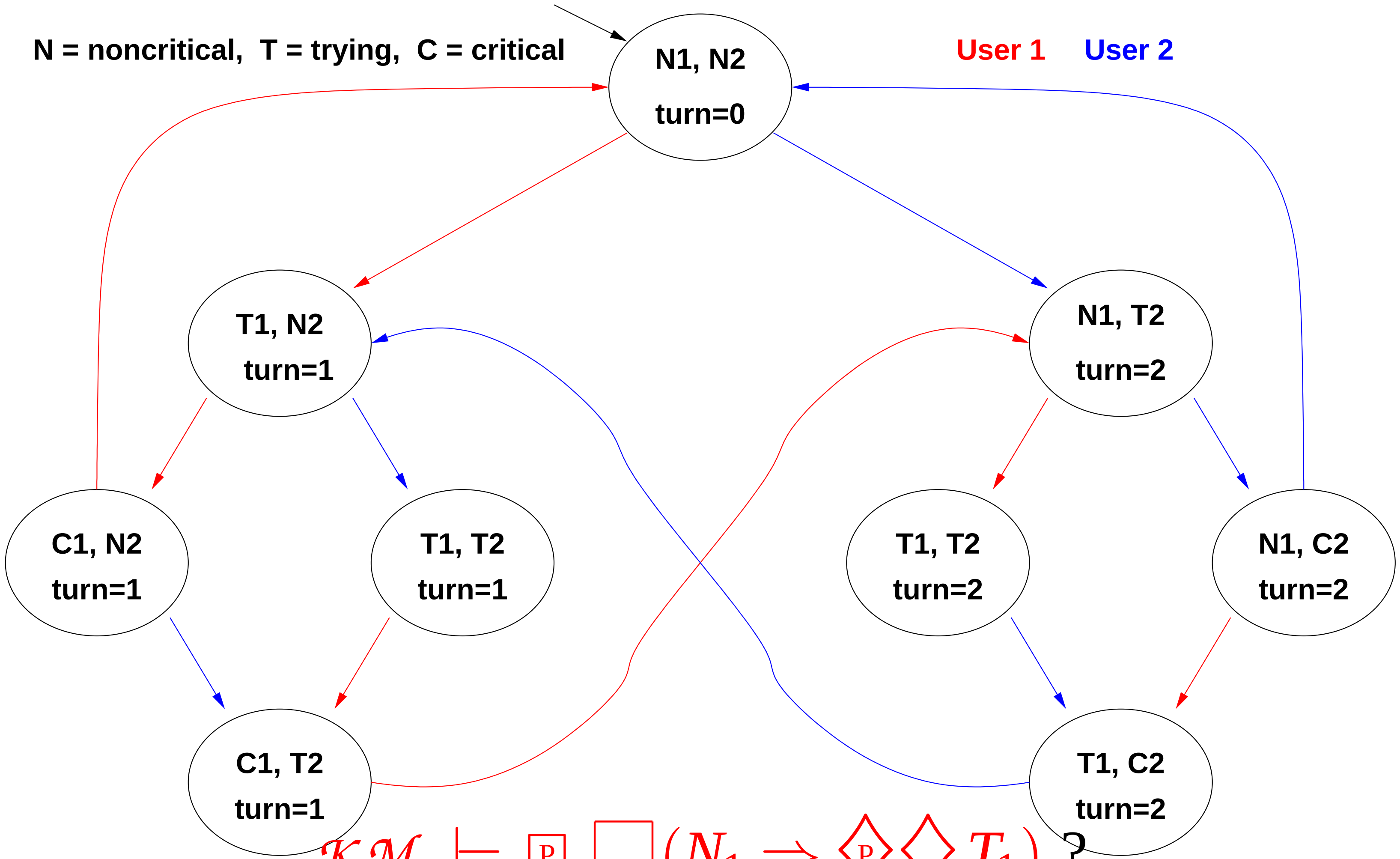
Example 3: Fairness



$\mathcal{KM} \models \boxed{\text{P}} \Box \boxed{\text{P}} \Diamond C_1$?

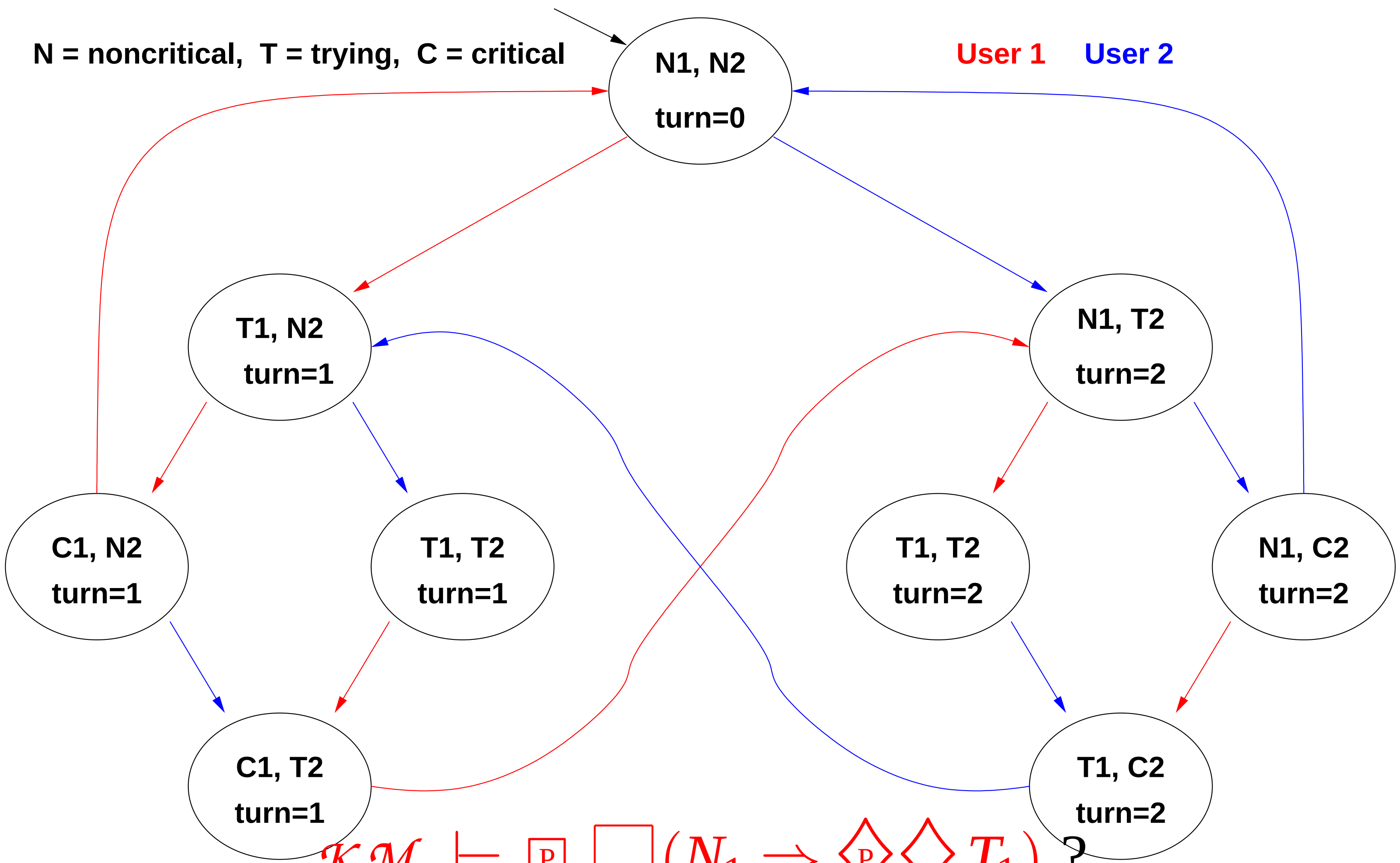
NO: e.g., in the initial state, there is the blue cyclic path in which C_1 never holds! (Same as $\Box\Diamond C_1$ in LTL)

Example 4: Non-Blocking



$$\mathcal{KM} \models \boxed{\mathrm{P}}\ \Box(N_1 \Rightarrow \Diamond_{\mathrm{P}} \Diamond T_1)\ ?$$

Example 4: Non-Blocking



$\mathcal{KM} \models \boxed{\text{P}}\,\Box(N_1 \Rightarrow \Diamond\!\!\!\!\text{P}\;\Diamond T_1)$?

YES: from each state where N_1 holds there is a path leading to a state where T_1 holds. (No corresponding LTL formulas)

Summary

- Computation Tree Logic: Intuitions.
- CTL: Syntax and Semantics.
- CTL in Computer Science.
- CTL and Model Checking: Examples.
- CTL Vs. LTL.
- CTL*.

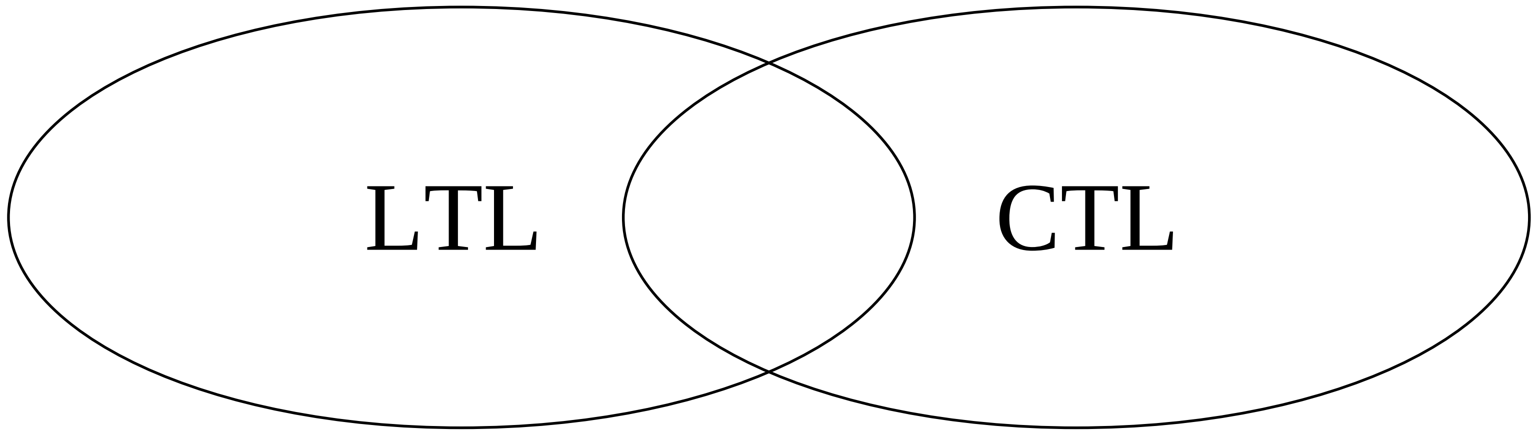
LTL Vs. CTL: Expressiveness

- Many CTL formulas cannot be expressed in LTL (e.g., those containing paths quantified existentially) E.g., $\boxed{\text{P}}\ \Box(N_1 \Rightarrow \Diamond\!\!\!\!\!\!{\scriptstyle\text{P}}\ \ \Diamond T_1)$
- Many LTL formulas cannot be expressed in CTL E.g., $\Box\Diamond T_1 \Rightarrow \Box\Diamond C_1$ (Strong Fairness in LTL) i.e, formulas that select a *range* of paths with a property ($\Diamond p \Rightarrow \Diamond q$ Vs. $\boxed{\text{P}}\ \Box(p \Rightarrow \boxed{\text{P}}\,\Diamond q)$)
- Some formluas can be expressed both in LTL and in CTL (typically LTL formulas with operators of nesting depth 1) E.g., $\Box\neg(C_1 \wedge C_2)$, $\Diamond C_1$, $\Box(T_1 \Rightarrow \Diamond C_1)$, $\Box\Diamond C_1$

LTL Vs. CTL: Expressiveness (Cont.)

CTL and LTL have incomparable expressive power.

The choice between LTL and CTL depends on the application and the personal preferences.



Summary

- Computation Tree Logic: Intuitions.
- CTL: Syntax and Semantics.
- CTL in Computer Science.
- CTL and Model Checking: Examples.
- CTL Vs. LTL.
- CTL*.

The Computation Tree Logic CTL*

- CTL* is a logic that combines the expressive power of LTL and CTL.
- Temporal operators can be applied without any constraints.
- $\boxed{\scriptstyle P}\,(\bigcirc\varphi \vee \bigcirc\bigcirc\varphi)$.
  Along all paths, φ is true in the next state or the next two steps.
- $\Diamond\kern-0.7em{\scriptstyle P}\;\,(\Box\Diamond\varphi)$.
  There is a path along which φ is infinitely often true.

CTL*: Syntax

Countable set Σ of atomic propositions: $p, q, \ldots$ we distinguish between *States Formulas* (evaluated on states):

$$\begin{array}{lll} \varphi, \psi & \longrightarrow & p \mid \top \mid \bot \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \\ & & \boxed{\mathrm{P}}\, \alpha \mid \Diamond_{\mathrm{P}}\, \alpha \end{array}$$

and *Path Formulas* (evaluated on paths):

$$\begin{array}{lll} \alpha, \beta & \longrightarrow & \varphi \mid \\ & & \neg\alpha \mid \alpha \wedge \beta \mid \alpha \vee \beta \mid \\ & & \bigcirc\alpha \mid \Box\alpha \mid \Diamond\alpha \mid (\alpha\, \mathcal{U}\, \beta) \end{array}$$

The set of CTL* formulas $\textsc{Form}$ is the set of state formulas.

CTL* Semantics: State Formulas

We start by defining when an atomic proposition is true at a state "s_0"

$$\mathcal{KM}, s_0 \models p \quad \textbf{iff} \quad p \in L(s_0) \qquad (\text{for } p \in \Sigma)$$

The semantics for *State Formulas* is the following where $\pi = (s_0, s_1, \ldots)$ is a generic path outgoing from state s_0:

$$\mathcal{KM}, s_0 \models \neg\varphi \quad \textbf{iff} \quad \mathcal{KM}, s_0 \not\models \varphi$$

$$\mathcal{KM}, s_0 \models \varphi \wedge \psi \quad \textbf{iff} \quad \mathcal{KM}, s_0 \models \varphi \text{ and } \mathcal{KM}, s_0 \models \psi$$

$$\mathcal{KM}, s_0 \models \varphi \vee \psi \quad \textbf{iff} \quad \mathcal{KM}, s_0 \models \varphi \text{ or } \mathcal{KM}, s_0 \models \psi$$

$$\mathcal{KM}, s_0 \models \Diamond_{P}\, \alpha \quad \textbf{iff} \quad \exists\pi = (s_0, s_1, \ldots) \text{ such that } \mathcal{KM}, \pi \models \alpha$$

$$\mathcal{KM}, s_0 \models \boxed{P}\, \alpha \quad \textbf{iff} \quad \forall\pi = (s_0, s_1, \ldots) \text{ then } \mathcal{KM}, \pi \models \alpha$$

CTL* Semantics: Path Formulas

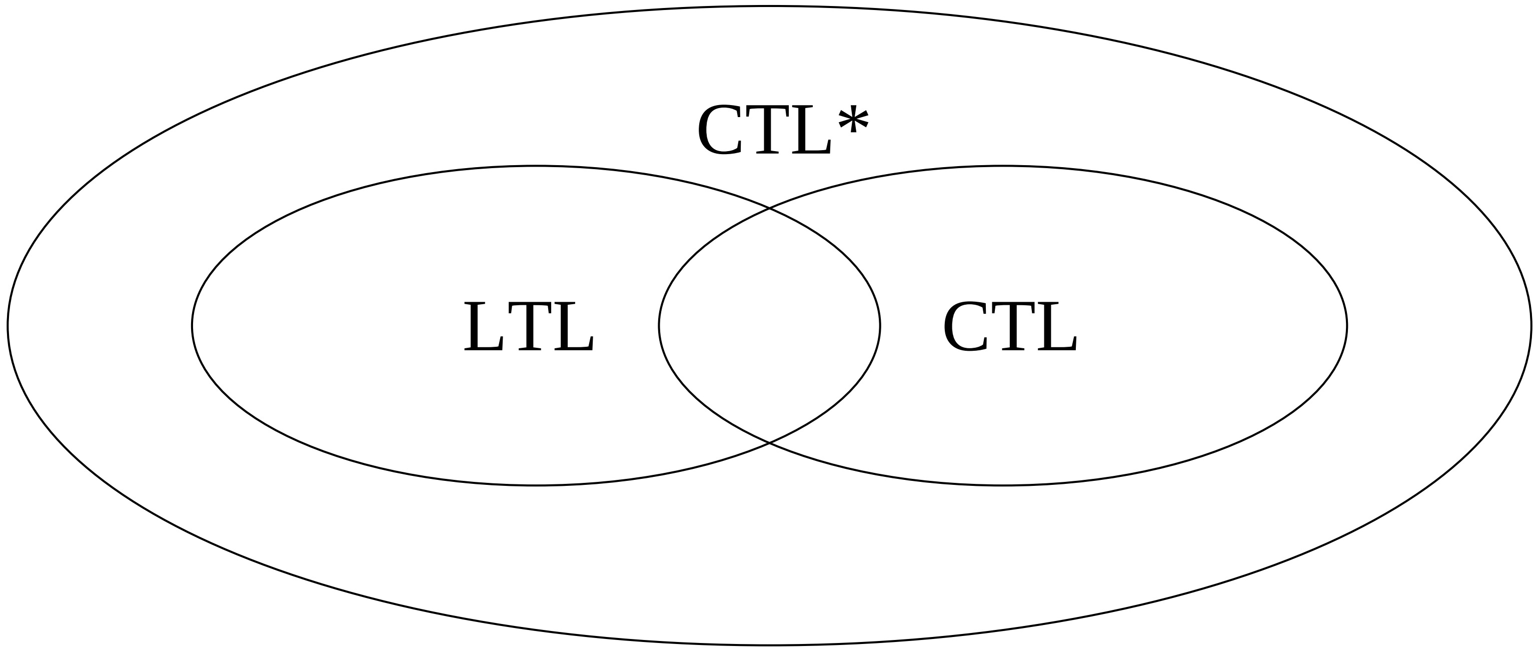
The semantics for *Path Formulas* is the following where $\pi = (s_0, s_1, \ldots)$ is a generic path outgoing from state s_0 and π^i denotes the suffix path $(s_i, s_{i+1}, \ldots)$:

$\mathcal{KM}, \pi \models \varphi$	**iff**	$\mathcal{KM}, s_0 \models \varphi$
$\mathcal{KM}, \pi \models \neg\alpha$	**iff**	$\mathcal{KM}, \pi \not\models \alpha$
$\mathcal{KM}, \pi \models \alpha \wedge \beta$	**iff**	$\mathcal{KM}, \pi \models \alpha$ and $\mathcal{KM}, \pi \models \beta$
$\mathcal{KM}, \pi \models \alpha \vee \beta$	**iff**	$\mathcal{KM}, \pi \models \alpha$ or $\mathcal{KM}, \pi \models \beta$
$\mathcal{KM}, \pi \models \Diamond\alpha$	**iff**	$\exists i \geq 0$ such that $\mathcal{KM}, \pi^i \models \alpha$
$\mathcal{KM}, \pi \models \Box\alpha$	**iff**	$\forall i \geq 0$ then $\mathcal{KM}, \pi^i \models \alpha$
$\mathcal{KM}, \pi \models \bigcirc\alpha$	**iff**	$\mathcal{KM}, \pi^1 \models \alpha$
$\mathcal{KM}, \pi \models \alpha\, \mathcal{U}\, \beta$	**iff**	$\exists i \geq 0$ such that $\mathcal{KM}, \pi^i \models \beta$ and $\forall j.(0 \leq j \leq i)$ then $\mathcal{KM}, \pi^j \models \alpha$

CTLs Vs LTL Vs CTL: Expressiveness

CTL* subsumes both CTL and LTL

- φ in CTL $\Longrightarrow$ φ in CTL* (e.g., $\boxed{\mathrm{P}}\,\Box(N_1 \Rightarrow \Diamond_{\mathrm{P}}\Diamond T_1)$)
- φ in LTL $\Longrightarrow$ $\boxed{\mathrm{P}}$ φ in CTL* (e.g., $\boxed{\mathrm{P}}\,(\Box\Diamond T_1 \Rightarrow \Box\Diamond C_1)$)
- LTL $\cup$ CTL $\subset$ CTL* (e.g., $\Diamond_{\mathrm{P}}(\Box\Diamond p \Rightarrow \Box\Diamond q)$)



CTL* Vs LTL Vs CTL: Complexity

The following Table shows the Computational Complexity of checking *Satisbiability*

Logic	**Complexity**
LTL	PSpace-Complete
CTL	ExpTime-Complete
CTL*	2ExpTime-Complete

CTL* Vs LTL Vs CTL: Complexity (Cont.)

The following Table shows the Computational Complexity of *Model Checking* (M.C.)

- Since M.C. has 2 inputs – the model, $\mathcal{M}$, and the formula, φ – we give two complexity measures.

Logic	**Complexity w.r.t.** $\lvert\varphi\rvert$	**Complexity w.r.t.** $\lvert\mathcal{M}\rvert$
LTL	PSpace-Complete	P (linear)
CTL	P-Complete	P (linear)
CTL*	PSpace-Complete	P (linear)

Summary of Lecture IV

- Computation Tree Logic: Intuitions.
- CTL: Syntax and Semantics.
- CTL in Computer Science.
- CTL and Model Checking: Examples.
- CTL Vs. LTL.
- CTL*.